ACTIONS OF SEMISIMPLE LIE GROUPS PRESERVING A DEGENERATE RIEMANNIAN METRIC

E. BEKKARA*, C. FRANCES†, AND A. ZEGHIB‡

Abstract. We prove a rigidity of the lightcone in Minkowski space. It is (essentially) the unique space endowed with a degenerate Riemannian metric, of lightlike type, and supporting an isometric non-proper action of a semi-simple Lie group.

1. Introduction

Our subject of study here is **lightlike** metrics on smooth manifolds. First, a lightlike scalar product on a vector space E is a symmetric bilinear form b which is positive but non-definite, and has exactly a 1-dimensional kernel. If E has dimension $1+n$, then, in some linear coordinates $(x^0, x^1, \dots, x^n)$, the associated quadratic form q can be written $q = (x^1)^2 + \dots + (x^n)^2$. Now, a lightlike metric h on a manifold M is a smooth tensor which is a lightlike scalar product on the tangent space of each point.

1.0.1. *Characteristic foliation.* The Kernel of h is a 1-dimensional sub-bundle $N \subset TM$, and thus determines a 1-dimensional foliation $\mathcal{N}$, called the **characteristic** (or null, normal, radical, isotropic...) foliation of h. By definition, any null curve (i.e. a curve with everywhere isotropic speed) of (M, h) trough x is contained in the *null leaf* $\mathcal{N}_x$. The (abstract) normal bundle of $\mathcal{N}$, i.e. the quotient TM/N is a Riemannian vector bundle. Conversely, a lightlike metric consists in giving a 1-dimensional foliation together with a Riemannian metric on its normal bundle.

1.1. **Major motivations.** Lightlike geometry appears naturally in a lot of geometric situations. We list now some natural examples motivating their study.

1.1.1. *Submanifolds of Lorentz manifolds.* Let M be a submanifold in a Lorentz manifold (V, g). The metric g is non-degenerate with signature $- + \dots +$. However, for a given $x \in M$, the restriction h_x of g to T_xM has not necessarily the same signature. Two easy stable situations are those where h_x is everywhere of Riemannian type (M is spacelike), or h_x is everywhere of Lorentzian type (M is timelike). In both cases, all the submanifold

Date: November 16, 2007.

1991 *Mathematics Subject Classification.* 53B30, 53C22, 53C50.

Key words and phrases. Lighlike metric, lightcone, isotropic direction.

* Partialy supported by the project CMEP 05 MDU 641B of the Tassili program.

theory valid in the Riemannian context generalizes: shape operator, Gauss and Codazzi equations...

The delicate situation is when h_x is degenerate for any x. Because the ambient metric has Lorentz signature, h_x is then lightlike as defined above. Unfortunately, by opposition to the previous cases, these lightlike submanifolds are generally "to poor" to generate a coherent extrinsic local metric differential geometry. Let us give examples of interesting lightlike submanifolds:

• *Horizons of domains of dependence and black holes.* Unfortunately, they have an essential disadvantage: their lower smoothness. One can believe that smooth horizons are sufficiently rigid to be classifiable (see for instance [6, 20, 13]).

• *Characteristic hypersurfaces of the wave equation.* There is a nice interpretation of lightlike hypersurfaces in terms of propagation of waves: a hypersurface is degenerate iff it is characteristic for the wave equation (on the ambient Lorentz space) [12]. These hypersurfaces enjoy the nice property that their null curves are geodesic in the ambient space (this is not true for submanifolds of higher codimension). However, no deeper study of their extrinsic geometry seems to be available in the literature.

• *Lightlike geodesic hypersurfaces.* They are characterized by the fact that their lightlike metrics are basic (see the example 1.2.1). They inherit a connection from the ambient space. See [8, 9, 22, 23], for their use in Lorentz dynamics.

• *Degenerate orbits of Lorentz isometric actions.* Let G be a Lie group acting isometrically on a Lorentz manifold (V, g). Then, any orbit which is lightlike at a point is lightlike everywhere, hence yields an embedded lightlike submanifold in V. The problem of understanding these lightlike orbits, and more generally degenerate invariant submanifolds, is essential when studying such isometric actions.

• *Terminology.* We believe that the choose of the word "lightlike" here is widely justified from the relationship between lightlike submanifolds and fields on one hand, and geometric as well as physical optics in general Relativity on the other hand (see for instance [21]). We also think this terminology is naturally adapted to our situation here, but less for the general situation of "singular pseudo-Riemannian" metrics (compare with [10, 17]).

1.1.2. *From submanifolds to intrinsic lightlike geometry.* In the last example given above, when restricting the action of the Lie group G to a lightlike orbit, we are led to study the isometric action of G on a lightlike submanifold in a Lorentzian manifold. In fact, one realizes that the submanifold structure is irrelevant in this problem, and the pertinent framework is that of isometric actions on abstract lightlike manifolds.

The main difficulty when dealing with this intrinsic formulation is that we loose the rigidity of the ambient action: as we will see below, the isometry group of a lightlike manifold can be infinitely dimensional.

1.2. **Two fundamental examples.** We give now two important examples of lightlike geometries, which are in some sense antagonistic.

1.2.1. *The most flexible example: transversally Riemannian flows.* The linear situation reduces to the case of $\mathbb{R}^{0,n}$, i.e $\mathbb{R}^{1+n}$ with coordinates $(x^0, x^1, \ldots, x^n)$, endowed with the lightlike quadratic form $q = (x^1)^2 + \ldots + (x^n)^2$.

We will denote its (linear) **orthogonal group** by $O(0,n)$ (this is somehow natural since reminiscent of the notation $O(1,n)$). We have:

$$O(0,n) = \left\{ \begin{pmatrix} \lambda & a_1 & \cdots & a_n \\ 0 & & & \\ \cdot & & A & \\ \cdot & & & \\ 0 & & & \end{pmatrix} \in GL(1+n, \mathbb{R}),\ A \in O(n) \right\}$$

It is naturally isomorphic to the affine similarity group $\mathbb{R} \times Euc_n = \mathbb{R}.O(n) \ltimes \mathbb{R}^n$ (here $Euc_n = O(n) \ltimes \mathbb{R}^n$ is the group of rigid motions of the Euclidian space of dimension n).

Let us now see $\mathbb{R}^{1+n}$ as a lightlike manifold. The group of its affine isometric transformations is $O(0,n) \ltimes \mathbb{R}^{1+n}$.

• Contrary to the non-degenerate case, *there is here a huge group (infinitely dimensional) of non-affine isometries.* Take any

$$\psi : (x^0, x^1, \ldots x^n) \mapsto (\psi_1(x^0, x^1, \ldots, x^n), \psi_2(x^1, .., x^n)),$$

where $\psi_2 \in Euc_n$, and $\psi_1 : \mathbb{R}^{n+1} \to \mathbb{R}$ is a smooth function with $\frac{\partial \psi_1}{\partial x^0} \neq 0$ (in order to get a diffeomorphism).

More generally, let (L, g) be a Riemannian manifold, and $M = \mathbb{R} \times L$ endowed with the lightlike metric $0 \oplus g$, that is the null foliation is given by the $\mathbb{R}$-factor, and the metric does not depend on the coordinate along it. Then, we have also here an infinitely dimensional group of isometric transformations given by: $\psi : (t, l) \in \mathbb{R} \times L \mapsto (\psi_1(t, l), \psi_2(l))$, where ψ_2 is an isometry of L, e.g. ψ_2 the identity map, and $\frac{\partial \psi_1}{\partial t} \neq 0$.

Conversely, assume that the lightlike metric (M, h) is such that there exists a non-singular vector field X tangent to the characteristic foliation, which flow preserves h (equivalently, the Lie derivative $L_X h = 0$). Then, locally, there is a metric splitting $M = \mathbb{R} \times L$ as above. Observe in fact that any vector field collinear to X will preserve h too, in other words any vector field orienting the characteristic foliation $\mathcal{N}$ preserves h. Let us call the lightlike metric **basic** in this case (they can also be naturally called, locally product, or stationary). This terminology is justified by the fact that h is the pull-back by the projection map $M \to L$ of the Riemannian metric on the basis L.

• Recall the classical notion from the geometric theory foliations: a 1-dimensional foliation $\mathcal{N}$ on a manifold M is transversally Riemannian (one then says $\mathcal{N}$ is a transversally Riemannian flow), if it is the characteristic foliation of some lightlike metric h on M, which is preserved by (local) vector fields tangent to $\mathcal{N}$. Therefore this data is strictly equivalent to giving a locally basic lightlike metric on M. Of course, the usual classical definition does not involve lightlike metrics.

There is a well developed theory of transversally Riemannian foliations, with sharp conclusions in the 1-dimensional case [7, 19]...

The isometry group of a basic lightlike metric contains at least all flows tangent to $\mathcal{N}$ which form an infinitely dimensional group (surely not so beautiful). However, these metrics are somehow tame, since, at least locally, the metric is encoded in an associated Riemannian one. Moreover, it t was proved by D. Kupeli [17] (and reproduced in many other places) that some kinf of Levi-Civita connection exists exactly if the lightlike metric is basic. The connection is never unique, and so enrichment of the structure is always in order. Actually, the most useful additional structure is that of a **screen**, mostly developed in [10], which allows to develop "calculus", and get sometimes invariant quantities (see for instance [3]). Nevertheless, there is generally no distinguished screen left invariant by the isometry group, so that this notion will not be helpful for us.

1.2.2. *The example of the Lightcone in Minkowski space.* We are going now to consider an opposite situation, where the isometry group is "big", though remaining finitely dimensional. Let $Min_{1,n}$ be the Minkowski space of dimension $1+n$, that is $\mathbb{R}^{1+n}$ endowed with the form $q = -x_0^2 + x_1^2 + \dots x_n^2$. The isotropic (positive) cone, or *lightcone*, Co^n is the set $\{q(x) = 0,\ x_0 > 0\}$. The metric induced by q on Co^n is lightlike. The group $O^+(1,n)$ (subgroup of $O(1,n)$ preserving the positive cone) acts isometrically on Co^n. This action is in fact transitive so that $Co^n = O^+(1,n)/Euc_{n-1}$ becomes a lightlike homogenous space, with isotropy group $Euc_{n-1} = O(n-1) \ltimes \mathbb{R}^{n-1}$, the group of rigid motions of the Euclidean space of dimension $n-1$.

A key observation is:

Theorem 1.1. *(Liouville Theorem for lightlike geometry) For, $n \geq 3$, any isometry of Co^n belongs to $O^+(1,n)$. In fact, this is true even locally for $n \geq 4$: any isometry between two connected open subsets of Co^n is the restriction of an element of $O^+(1,n)$.*
• *For $n = 3$, the group of local isometries is in one-to-one correspondence with the group of local conformal transformations of $\mathbf{S}^2$.*
• *For $n = 2$, there is no rigidity at all, even globally, since to any diffeomophism of the circle corresponds an isometry of Co^2.*

This theorem, which will be proved in §2, shows in particular that for $n \geq 3$, Co^n is a homogeneous lightlike manifold with isometry group $O(1,n)$.

Remark that for the sake of simplicity, we will often use the notation $O(1,n)$ to mean its identity component $SO_0(1,n)$, and generally any finite index subgroups of $O(1,n)$. Actually, to be precise, we can say that our geometric descriptions of objects are always given *up to a finite cover*.

It seems likely that being homogeneous and having a maximal isotropy $O(0,n-1)$ characterizes the flat case, i.e. $\mathbb{R}^{0,n-1}$, and having a maximal unimodular isotropy, i.e. Euc_{n-1}, characterizes the lightcone. In some sense the lightcone is the maximally symmetric non-flat lightlike space, analogous to spaces of constant non-zero curvature in the pseudo-Riemannian case.

1.3. **Statement of results.** The present article contains in particular detailed proofs of the results announced in [5]. Before giving the statements, let us recall that two (lightlike) metrics h and h' on a manifold M are said to be *homothetic* if $h = \lambda h'$, for a real $\lambda > 0$. A Lie group acts locally faithfully on M if the kernel of the action is a discrete subgroup.

One motivation of the present work was Theorem 1.6 of [9], that we state here as follows:

Theorem 1.2 ([9])**.** *Let G be a connected group with finite center, locally isomorphic to $O(1,n)$ or $O(2,n)$, $n \geq 3$. If G acts isometrically on a Lorentz manifold, and has a degenerate orbit with non-compact stabilizer, then G is locally isomorphic to $O(1,n)$, and the orbit is homothetic to the lightcone Co^n.*

Here, we prove an intrinsic version of this result:

Theorem 1.3. *Let G be a non-compact semi-simple Lie group with finite center acting locally faithfully, isometrically and* **non-properly** *on a lightlike manifold (M,h). Assume that G has no factor locally isomorphic to $SL(2,\mathbb{R})$. Then, looking if necessary at a finite cover of G:*

- *$G = H \times H'$, where H is locally isomorphic to $O(1,n)$.*
- *G has an orbit which is homothetic, up to a finite cover, to a metric product $Co^n \times N$, where N is a Riemannian H'-homogeneous manifold. The action of $H \times H'$ on $Co^n \times N$ is the product action.*

Using this theorem and working a little bit more, we can also handle the case where some factors of G are locally isometric to $SL(2,\mathbb{R})$, when the action is transitive. The following result can be thought as a converse to Theorem 1.1:

Corollary 1.4. *Let G be a non-compact semi-simple Lie group with finite center, acting locally faithfully, isometrically, transitively and non-properly on a lightlike manifold (M,h), i.e M is a homogeneous lightlike space G/H, with a non-compact isotropy group H. Then a finite cover of G is isomorphic to $O(1,d) \times H$, where $d \geq 2$ and H is semi-simple. The manifold M is, up to a finite cover, homothetic to a metric product $Co^m \times N$, where N is an H-homogeneous Riemannian space. Moreover $m = d$ when $d \geq 3$, and $m = 1$ or 2 when $d = 2$. The action of G on M is the product action*

The non-properness assumption is essential in the previous theorems. If one removes it, "everything becomes possible". Indeed, consider a Lie group L and a lightlike scalar product on its Lie algebra $\mathfrak{l}$. Translating it on L by left multiplication yields a lightlike metric on L, with isometry group containing L, acting by left translations.

It is quite surprising that kind of global rigidity theorems can be proved in the framework of lightlike metrics, which are not rigid geometric structures (see §1.2.1). Here, it is, in some sense, the algebraic assumption of semi-simplicity which makes the situation rigid. However, since any Lie algebra is a semi-direct product of a semi-simple and a solvable one, it is natural to start looking to actions of semi-simple Lie groups.

When the manifold M is compact, only one simple Lie group can act isometrically, as shows the:

Theorem 1.5. *Let G be a non-compact simple Lie group with finite center, acting isometrically on a compact lightlike manifold (M, h). Then G is a finite covering of $PSL(2, \mathbb{R})$, and all the orbit of G are closed, 1-dimensional, and lightlike.*

1.4. **The mixed signature case: sub-Lorentz metrics.** This notion will naturally modelize the situation of general submanifolds in Lorentz submanifolds. A *sub-Lorentz metric* g on M is a symmetric covariant 2 -tensor, which is at each point, a scalar product of either Lorentz, Euclidean, or lightlike type. The point is that we allow the type to vary over M. So, if (L, h) is a Lorentz manifold, and M a submanifold of L, then the restriction on h on M is a sub-Lorentz metric (this fact raises the inverse problem, i.e. isometric embedding of sub-Lorentz metrics in Lorentz manifolds). We think it is worthwhile to investigate the geometry of these natural and rich structures (see for instance [18] for a research of normal forms of these metrics in dimension 2).

We restrict our investigation here to an adaptation of our lightlike results to this sub-Lorentz situation.

1.4.1. *Lorentz dynamics.* Recall the three fundamental examples of Lorentz manifolds having an isometry group which acts non-properly. They are just the universal spaces of constant curvature:

(1) The minkowski space: $Min_{1,n-1} = O(1, n-1) \ltimes \mathbb{R}^n / O(1, n-1)$
(2) The de Sitter space $dS_n = O(1, n)/O(1, n-1)$
(3) The anti de Sitter space $AdS_n = O(2, n-1)/O(1, n-1)$

In the case of Minkowski space, the isometry group is not semi-simple.

The Lorentz and lightlike dynamics are unified in the following statement:

Theorem 1.6. *Let G be a semi-simple group with finite center , no compact factor and no local factor isomorphic to $SL(2, \mathbb{R})$, acting isometrically non-properly on a sub-Lorentz manifold M. Then, up to a finite cover, G has a* *factor G' isomorphic to $O(1,n)$ or $O(2,n)$ and having some orbit homothetic to dS_n, AdS_n or Co^n.*

2. Preliminaries

2.1. **Proof of Theorem 1.1.** The metric on Co^n is just the metric $0 \oplus e^{2t} g_{\mathbf{S}^{n-1}}$ on $\mathbb{R} \times \mathbf{S}^{n-1}$. An isometry f of Co^n is of the form $(t,x) \mapsto (\lambda(t,x), \phi(x))$. A simple calculation proves that f is isometric iff:

$$\phi^* g_{\mathbf{S}^{n-1}} = e^{2(t-\lambda(t,x))} g_{\mathbf{S}^{n-1}}$$

So, any local isometry of Co^n is of the form $(t,x) \mapsto (t-\mu(x), \phi(x))$, with ϕ a local conformal transformation of the sphere satisfying $\phi^* g_{\mathbf{S}^{n-1}} = e^{2\mu} g_{\mathbf{S}^{n-1}}$. Thus, the different rigidity phenomena are just consequences of classical analogous rigidity results for conformal transformations on the sphere. □

2.2. $SL(2,\mathbb{R})$**-homogeneous spaces.** Understanding these spaces is worthwhile in our context, since one can take advantage of restricting the G-action to small simpler groups, e.g. $SL(2,\mathbb{R})$ or a finite cover, which always exist in semi-simple Lie groups.

2.2.1. *Notations.* Let $SL(2,\mathbb{R})$ be the Lie group of 2×2-matrices with determinant 1. It is known that any one parameter subgroup of $SL(2,\mathbb{R})$ is conjugate to one of the following:

$$A^+ = \left\{ \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix},\ t \in \mathbb{R} \right\}, N = \left\{ \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix},\ t \in \mathbb{R} \right\}$$

$$\text{or}\quad K^+ = \left\{ \begin{pmatrix} \sin t & -\cos t \\ \cos t & \sin t \end{pmatrix},\ t \in \mathbb{R} \right\}.$$

The corresponding derivatives of A^+ and N at the identity are

$$X = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \text{ and } Y = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Together with $Z = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, X and Y span the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ and satisfy the bracket relations:

$$[X,Y] = 2Y, \quad [X,Z] = -2Z \text{ and } \quad [Y,Z] = X.$$

As usual, we denote by A (resp. K), the subgroup generated by $A^+, -A^+$ (resp. $K^+, -K^+$).

Let $Aff(\mathbb{R})$ be the subgroup of upper triangular matrices:

$$Aff(\mathbb{R}) = A.N = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \in SL(2,\mathbb{R}) \right\}$$

and $\mathfrak{aff}(\mathbb{R})$ its Lie algebra.

Non-connected 1-dimensional subgroups of $Aff(\mathbb{R})$ can be constructed as follows. Let Γ_0 be a cyclic subgroup of A generated by an element $\gamma \in A$.

The semi-direct product $\Gamma_0 \ltimes N$ is a closed subgroup. Conversely, any closed 1-dimensional non-connected subgroup of $Aff(\mathbb{R})$ is obtained like this.

Finally, recall $PSL(2,\mathbb{R}) = SL(2,\mathbb{R})/_{\{\pm Id\}}$.

The "classical" classification of the $SL(2,\mathbb{R})$-homogenous spaces, allows one to recognize the lightlike ones.

Proposition 2.1. *(Classification of $SL(2,\mathbb{R})$-homogeneous spaces)*

(1) *Any $SL(2,\mathbb{R})$-homogenous space is isomorphic to one of the following:*
   (a) *The circle $S^1 = SL(2,\mathbb{R})/_{Aff(\mathbb{R})}$, endowed with its natural projective structure.*
   (b) *The hyperbolic plane $= SL(2,\mathbb{R})/_K$, with its Riemannian metric of constant negative curvature.*
   (c) *The affine punctured plane: $\mathbb{R}^2 \setminus \{0\} = SL(2,\mathbb{R})/_N$, equipped with an affine flat connection, together with a lightlike metric.*
   (d) *A Hopf affine torus $\mathbb{R}^2 \setminus \{0\}/_{\{x\sim ax\}} = SL(2,\mathbb{R})/_{\Gamma_0.N}$, endowed with a flat projective structure.*
   (e) *A space $SL(2,\mathbb{R})/_\Gamma$, where Γ is a discrete subgroup of $SL(2,\mathbb{R})$. It is locally an Anti de Sitter space, i.e. a Lorentz manifold with negative constant curvature.*

(2) *Up to homothety, the unique lightlike $SL(2,\mathbb{R})$-homogenous spaces having a non-compact isotropy are:*
   (a) *The lightcone Co^1, i.e the circle S^1 endowed with the null metric.*
   (b) *The lightcone Co^2, namely $\mathbb{R}^2 \setminus \{0\}$, endowed with the lightlike metric $d\theta^2$, where $\mathbb{R}^2 \setminus \{0\}$ is parameterized by the polar coordinates (r, θ).*

Proof. The proof of the first part is standard; we just give details in the lightlike case.

Let Σ be be an $SL(2,\mathbb{R})$-homogeneous space of dimension ≥ 2 i.e $\Sigma \cong SL(2,\mathbb{R})/_H$, where H is the stabilizer of some $p \in \Sigma$ and conjugated, as showed above, to one of the following subgroups: $K, N, \Gamma_0 N$ and Γ. Let $\mathfrak{h}$ be the Lie algebra of H. Considering the isotropy representation

$$\rho_H : \quad H \quad \longrightarrow \quad T_p(\Sigma) = \mathfrak{g}/\mathfrak{h}$$

one observes that when $H = K$ or $\Gamma_0 N$, $\rho_H(H)$ is not conjugated to a subgroup of $O(0,1)$. Now, if $H = \Gamma$, then $\rho_H(\Gamma)$ is conjugated to a subgroup of $O(1,2)$. This is just because the Killing form on $\mathfrak{sl}(2,\mathbb{R})$ has Lorentz signature. If moreover $\rho_H(\Gamma)$ is conjugated to a subgroup of $O(0,2)$, then $\rho_H(\Gamma)$ has to be finite. Since the Kernel of the adjoint representation of $SL(2,\mathbb{R})$ is finite, we get that Γ is finite. Therefore the unique lightlike $SL(2,\mathbb{R})$-homogeneous space of dimension ≥ 2 with non-compact isotropy is $\mathbb{R}^2 \setminus \{0\}$.

In order to check that the lightlike metric has the form $\alpha d\theta^2$ (for some $\alpha \in \mathbb{R}_+^*$), one argues as follows. At $p = (1,0)$, the vector X is the unique non-trivial eigenspace of ρ_N, and thus the orbit of p by the flow ϕ_X^t must coincide with the null leaf $\mathcal{N}_{(1,0)}$, which is therefore a radial half-line. The other null leaves are also radial, since they are images of $\mathcal{N}_{(1,0)}$ by the $SL(2,\mathbb{R})$-action. By homogeneity, the metric must have the form $\alpha d\theta^2$. $\square$

Remark 2.2. *Proposition 2.1 is a special case of Theorem 1.3 where $G = O(1,2)$.*

For a latter use, let us state the following fact, which follows directly from the previous description of the lightlike surface $\mathbb{R}^2 \setminus \{0\}$.

Fact 2.3. *If Y is isotropic at some $p \in \mathbb{R}^2 \setminus \{0\}$, then Y vanishes at p and X is isotropic at p.*

2.3. **Generalities on semi-simple groups; notations.** [See for instance [14]] Let G be a semi-simple group acting isometrically on (M,h). This means that we have a smooth homomorphism $\rho : G \to Diff^\infty(M)$, such that for every $g \in G$, $\rho(g)$ acts as an isometry for h (i.e $\rho(g)^*h = h$). Let $\mathfrak{g}$ be the Lie algebra of G. For any X in $\mathfrak{g}$, we will generally use the notation ϕ_X^t instead of $\rho(\exp(tX))$. By a slight abuse of language, we will also denote by X the vector field of M generated by the flow ϕ_X^t.

We get, for every $p \in M$, a homomorphism $\lambda_p : \mathfrak{g} \to T_pM$, defined by $\lambda_p(X) = X_p$. The flow ϕ_X^t stabilizes p iff $X_p = 0$, and we denote by $\mathfrak{g}_p$ the Lie algebra of the stabilizer of p.

We say that $X \in \mathfrak{g}$ is *lightlike at $p \in M$* (or *isotropic*) (resp. *spacelike*) if $h_p(X_p, X_p) = 0$ (resp. $h_p(X_p, X_p) > 0$).

We denote by $\mathfrak{s}_p$ the subspace of all vectors of $\mathfrak{g}$ which are isotropic at $p \in M$.

Let O be a lightlike G-orbit of some $p \in M$, that is $O \cong G/G_p$, where G_p is the stabilizer of p. The tangent space T_pO is identified by λ_p to the quotient $\mathfrak{g}/\mathfrak{g}_p$. In fact the isotropy representation on T_pO is equivalent to the adjoint representation Ad of G_p on $\mathfrak{g}/\mathfrak{g}_p$. In particular G_p is mapped, up to conjugacy, to a subgroup of $O(0,n)$.

Similarly, the Euclidian space $T_pO/\mathcal{N}_p$ is identified to $\mathfrak{g}/\mathfrak{s}_p$, where $Ad : G_p \longrightarrow GL(\mathfrak{g}/\mathfrak{s}_p)$ preserves a positive inner product on $\mathfrak{g}/\mathfrak{s}_p$, so that G_p acts on $\mathfrak{g}/\mathfrak{s}_p$ by orthogonal matrices. In particular, if we consider the tangent representation: $ad : \mathfrak{g}_p \longrightarrow End(\mathfrak{g}/\mathfrak{s}_p)$, the Lie subalgebra $\mathfrak{g}_p$ acts by skew symmetric matrices on $\mathfrak{g}/\mathfrak{s}_p$. We will use the same notation for the elements of the quotients $\mathfrak{g}/\mathfrak{g}_p$ and $\mathfrak{g}/\mathfrak{s}_p$ and their representatives in the Lie algebra $\mathfrak{g}$.

We fix once for all a Cartan involution Θ on the Lie algebra $\mathfrak{g}$. This yields a *Cartan decomposition* $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, $\mathfrak{k}$ (resp. $\mathfrak{p}$) being the eigenspace of Θ associated with the eigenvalue $+1$ (resp. -1).

We choose $\mathfrak{a}$, a maximal abelian subalgebra of $\mathfrak{p}$, and $\mathfrak{m}$ the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$. This choice yields a *rootspace decomposition* of $\mathfrak{g}$, namely there is a finite family $\Sigma^+ = \{\alpha_1, ..., \alpha_s\}$ of nonzero elements of $\mathfrak{a}^*$, such that $\mathfrak{g} = \oplus_{\alpha\in\Sigma^+}\mathfrak{g}_{-\alpha} \oplus \mathfrak{g}_0 \oplus \oplus_{\alpha\in\Sigma^+}\mathfrak{g}_\alpha$. For every $X \in \mathfrak{a}$, $ad(X)(Y) = \alpha(X)Y$, as soon as $Y \in \mathfrak{g}_\alpha$. The Lie subalgebra $\mathfrak{g}_0$ is in the kernel of $ad(X)$, for every $X \in \mathfrak{a}$ and splits as a sum: $\mathfrak{g}_0 = \mathfrak{a} \oplus \mathfrak{m}$.

The positive Weyl chamber $\mathfrak{a}^+ \subset \mathfrak{a}$, contains those $X \in \mathfrak{a}$, such that $\alpha(X) \geq 0$, for all $\alpha \in \Sigma^+$. Its image by the exponential map is denoted by A^+. Let $\Sigma^- = \{-\alpha_1, ..., -\alpha_s\}$.

The stable subalgebra (for $\mathfrak{a}$) $W^s = \oplus_{\alpha\in\Sigma^-}\mathfrak{g}_\alpha$, and the unstable one $W^u = \oplus_{\alpha\in\Sigma^+}\mathfrak{g}_\alpha$ are both nilpotent subalgebras of $\mathfrak{g}$, mapped diffeomorphically by the exponential map of $\mathfrak{g}$ onto two subgroups $N^+ \subset G$, and $N^- \subset G$.

Given $X \in \mathfrak{a}$, its **stable** algebra is $W^s_X = \oplus_{\alpha(X)<0}\mathfrak{g}_\alpha$, and its **unstable** algebra is $W^u_X = \oplus_{\alpha(X)>0}\mathfrak{g}_\alpha$.

Let us prove now a lemma which will be useful in the sequel:

Lemma 2.4. *The subalgebra W^s_X has the following properties:*

(1) $[\mathfrak{g}, W^s_X \cap \mathfrak{g}_p] \subset \mathfrak{s}_p$
(2) $[\mathfrak{s}_p, W^s_X \cap \mathfrak{g}_p] \subset \mathfrak{g}_p$

Proof. Let $Y \in W^s_X \cap \mathfrak{g}_p$

(1) Since $Y \in \mathfrak{g}_p$, ad_Y acts on $\mathfrak{g}/\mathfrak{s}_p$ by a skew symetric endomorphism, which is moreover nilpotent since $Y \in W^s_X$. Hence ad_Y acts by the null endomorphism on $\mathfrak{g}/\mathfrak{s}_p$, which means that ad_Y maps $\mathfrak{g}$ to $\mathfrak{s}_p$.
(2) ad_Y acts as a nilpotent endomorphism of $\mathfrak{g}/\mathfrak{g}_p$ (identified with the tangent space), and has $\mathfrak{s}_p/\mathfrak{g}_p$ (identified to the isotropic direction) as a 1-dimensional eigenspace. By nilpotency, the action on it is trivial, i.e ad_Y maps $\mathfrak{s}_p$ into $\mathfrak{g}_p$.

□

Finally, recall that a semi-simple Lie group of finite center admits a Cartan decomposition $G = KAK$, where K is a maximal compact subgroup of G.

2.4. **Non-proper actions.** (See for instance [15] for a recent survey about these notions).

Definition 2.5. *Let G act on M. A sequence (p_k) is* **non-escaping** *if there is a sequence of transformations $g_k \in G$, such that, both (p_k) and $(q_k) = (g_k(p_k))$ lie in a compact subset of M, but (g_k) tends to ∞ in G, i.e. leaves any compact of G.*

– The sequence (g_k) is called a "return sequence" *for (p_k).*

– In the sequel, we will sometimes assume that (p_k) and (q_k) converge to p and q in M.

One says that the group G acts non-properly if it admits a non-escaping sequence.

A nice criterion for actions of semi-simple Lie groups of finite center to be non-proper, is the next:

Lemma 2.6. *Let G be a non-compact semi-simple group with finite center. Then G acts non-properly iff any Cartan subgroup A acts non-properly.*

Proof. G admits a Cartan decomposition KAK, where K is compact. Let (p_k) be a non-escaping sequence of the G-action, and (g_k) its return sequence. Write $g_k = l_k a_k r_k \in KAK$. Then, $p'_k = r_k(p_k)$ is a non-escaping sequence for the A-action, with associated return sequence (a_k). Obviously (a_k) goes to infinity in A since (g_k) goes to infinity in G. □

3. A Key fact on the stable space

Here we state a crucial ingredient for the proofs of all our theorems. In all what follows, G is a non-compact semi-simple Lie group with finite center acting locally faithfully, non-properly and isometrically on a lightlike manifold (M, h). The main result of this section is:

Proposition 3.1. *If no factor of G is locally isomorphic to $SL(2, \mathbb{R})$, there exists a Cartan subalgebra $\mathfrak{a}_0$ such that for some $X_0 \in \mathfrak{a}_0$ and $p_0 \in M$, both X_0 and its stable algebra $W^s_{X_0}$ are isotropic at p_0.*

3.1. **Starting fact.** The non-properness of the action of G leads to a fundamental fact, already observed in [16] for Lorentzian metrics, which is the existence of $p \in M$ and $X \in \mathfrak{a}$ such that W^s_X is isotropic at p. Let us recall its proof.

Proposition 3.2. [16] *Let $\mathfrak{a}$ be a Cartan subalgebra of $\mathfrak{g}$.*

(1) *If the flow of $X \in \mathfrak{a}$ acts non-properly, and $p_k \to p$ is a non-escaping sequence for the action of ϕ^t_X, then the stable space W^s_X is isotropic at p.*
(2) *More generally, if $p_k \to p$ is a non-escaping sequence for the A-action, then there exists $X \in \mathfrak{a}$ such that W^s_X is isotropic at $p \in M$.*

Proof. (1) Denote $\phi^t_X = exptX$ the flow of X, and let (t_k) be a return time sequence for (p_k), i.e. $\phi^{t_k}_X$ is a return sequence of p_k, what means that $q_k = \phi^{t_k}_X(p_k)$ stay in a compact subset of M.

Let $Y \in \mathfrak{g}_\alpha$, then $[X, Y] = \alpha(X)Y$, hence for any $x \in M$, $D_x\phi^t_X Y_x = e^{t\alpha(X)}Y_{\phi^t_X(x)}$. Assume that $\alpha(X) < 0$, then:

$$h_{p_k}(Y_{p_k}, Y_{p_k}) = h_{q_k}(D_{p_k}\phi^{t_k}_X(Y_{p_k}), D_{p_k}\phi^{t_k}_X(Y_{p_k})) = e^{2t_k\alpha(X)}h_{q_k}(Y_{q_k}, Y_{q_k})$$

On the left hand side, passing to the limit yields $h_p(Y_p, Y_p)$.

On the right hand side, since (q_k) lie in a compact set, $h_{q_k}(Y_{q_k}, Y_{q_k})$ is bounded. Therefore, since $\alpha(X) < 0$, this right hand term tends to 0, yielding $h_p(Y_p, Y_p) = 0$. This proves that W^s_X is isotropic at p.

(2) Let (X_k) be a sequence in $\mathfrak{a}$ such that $exp(X_k)$ is a return sequence for (p_k). Let $\|.\|$ be a Euclidian norm on $\mathfrak{a}$ and, considering if necessary a subsequence, assume $(\frac{X_k}{\|X_k\|})$ converges to some $X \in \mathfrak{a}$. As above, one proves that that W^s_X is isotropic at p.

□

Remark 3.3. *This result is nothing but a generalization of the linear (punctual) easy fact: If a matrix A preserves a lightlike scalar product, then its corresponding stable and unstable spaces are isotropic. In our particular case, if $X \in \mathfrak{a} \cap \mathfrak{g}_p$ (i.e. X stabilizes p), then both of W^s_X and W^u_X are isotropic at p.*

3.2. **Proof of Proposition 3.1.** The proof follows from several observations. The simplest one is that for lightlike metrics (in contrast with the Lorentz case), the isotropic direction is unique on each tangent space T_pM. Furthermore, it coincides with the non-trivial eigenspace (if any) of any infinitesimal isometry fixing p. The hypothesis made in Proposition 3.1, that G has no factor locally isomorphic to $SL(2,\mathbb{R})$ will be only used in Lemma 3.7.

Lemma 3.4. *For any $p \in M$, the subspace of isotropic vectors $\mathfrak{s}_p$ is a Lie subalgebra of $\mathfrak{g}$.*

Proof. Let $X, Y \in \mathfrak{s}_p$, and let ϕ^t_X be the isometric flow generated by X on M, then:

$$[X,Y]_p = \lim_{t\to 0} \frac{1}{t}[d\phi^{-t}_X(Y_{\phi^t_X(p)}) - Y_p]$$

since X, Y are isotropic at p, their integral curves at p are supported by the null leaf $\mathcal{N}_p$, and thus $Y_{\phi^t_X(p)}$ is isotropic. Since ϕ^{-t}_X is an isometry, $d\phi^{-t}_X(Y_{\phi^t_X(p)})$ is also isotropic. □

Lemma 3.5. *G stabilizes no $p \in M$.*

Proof. Suppose by contradiction that G stabilizes $p \in M$, then G acts on T_pM by: $\rho : g \mapsto d_pg \in GL(T_pM)$. Since G preserves the lightlike scalar product h_p, it is mapped by ρ into a subgroup of $O(0,n)$. Thus, at the level of Lie algebras, we get a homomorphism $d\rho : \mathfrak{g} \to \mathfrak{o}(0,n)$. Now, we prove:

Sublemma 3.6. *Any homomorphism from $\mathfrak{g}$ to $\mathfrak{o}(0,n)$ is trivial.*

Proof. Without loss of generality, we can suppose that $\mathfrak{g}$ is simple. Let λ be a homomorphism from $\mathfrak{g}$ to $\mathfrak{o}(0,n)$, π the projection from $\mathfrak{o}(0,n)$ to $\mathfrak{o}(n)$. Consider the homomorphism $\lambda \circ \pi : \mathfrak{g} \longrightarrow \mathfrak{o}(n)$. Since $\mathfrak{g}$ is simple and noncompact, it has no non-trivial homomorphism into the Lie algebra of a compact group; this implies that $\lambda \circ \pi$ is trivial. So, $\mathfrak{g}$ is mapped by λ into the kernel $\mathfrak{g}_0$ of π, that is the algebra of matrices of the form

$$\begin{pmatrix} \mu & x_1 & \dots & x_n \\ 0 & & & \\ . & & 0 & \\ . & & & \\ 0 & & & \end{pmatrix}$$. Since $\mathfrak{g}_0$ is solvable and $\mathfrak{g}$ is simple, we conclude that λ is trivial. □

As a corollary, the ρ-image of any connected compact subgroup $K \subset G$ is trivial. However such K preserves a Riemannian metric. But on a connected manifold M, a Riemannian isometry which fixes a point and has a trivial derivative at this point, is the identity on M. This is easily seen since in the neighbourhood of any fixed point, a Riemannian isometry is linearized thanks to the exponential map. Hence K acts trivially on M, and therefore G does not act faithfully, which contradicts our hypothesis and completes the proof of our lemma. □

Lemma 3.7. *If G has no factor locally isomorphic to $SL(2,\mathbb{R})$, then no Cartan subalgebra $\mathfrak{a}$ meets the stabilizer subalgebra: $\mathfrak{a} \cap \mathfrak{g}_p = \{0\}$ for any $p \in M$.*

Proof. Assume by contradiction that, $\mathfrak{a} \cap \mathfrak{g}_p \neq \{0\}$, and let us take $X \neq 0$ in this intersection. Apply Remark 3.3 to X to get that the subspaces W^s_X and W^u_X are both isotropic at p. It is a general fact that $\mathfrak{n}$, the Lie subalgebra generated by W^s_X and W^u_X, is an ideal of $\mathfrak{g}$ (see for instance [16]), and it is in particular a factor of $\mathfrak{g}$. It acts on the 1-manifold $\mathcal{N}_p$. This action is faithful, otherwise its kernel $\mathfrak{s}$ would be the Lie algebra of a semi-simple group $S \subset G$, which would have fixed points on M, in contradiction with Lemma 3.5. Now, the only the semi-simple algebra acting faithfully on a 1-manifold is $\mathfrak{sl}(2,\mathbb{R})$. This contradicts our hypothesis that $\mathfrak{g}$ has no such factor. □

Lemma 3.8. *Let H be a Lie group having $\mathfrak{sl}(2,\mathbb{R})$ as a Lie algebra.*

(1) *If H is linear, then it is isomorphic to either $SL(2,\mathbb{R})$ or $PSL(2,\mathbb{R})$.*
(2) *If H is a subgroup of a Lie group G with finite center, then it is a finite covering of $PSL(2,\mathbb{R})$.*

Proof. The point is that all the representations of the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ integrate to actions of the group $SL(2,\mathbb{R})$ itself (and not merely its universal cover). Indeed, the classical classification asserts that all the irreducible representations are isomorphic to symmetric powers of the standard representation, or equivalently to representations on spaces of homogeneous polynomials of a given degree, in two variables x and y (see for instance [14]). Clearly, $SL(2,\mathbb{R})$ acts on these polynomials, and $PSL(2,\mathbb{R})$ acts iff the degree is even. For the last point, observe that the adjoint representation of G has finite Kernel. □

End of the proof of Proposition 3.1. From Proposition 3.2, there exist $X \in \mathfrak{a}$ and $p \in M$, such that W^s_X is isotropic at p. Since $\mathfrak{g}$ has no local factor isomorphic to $\mathfrak{sl}(2,\mathbb{R})$, we have $\dim W^s_X > 1$ (otherwise the subalgebra $\mathfrak{a} \oplus$

$\Sigma_{\alpha(X)\geq 0}\mathfrak{g}_\alpha$ is supplementary to W^s_X and would have codimension 1, and therefore $\mathfrak{g}$ acts on a 1-manifold). For a lightlike metric, an isotropic space has dimension at most 1, so that the evaluation of W^s_X at p has at most dimension 1, and thus W^s_X contains at least a non-zero vector Y_0 vanishing at p.

By the Jacobson-Morozov Theorem (see [14]), the nilpotent element Y_0 belongs to some subalgebra $\mathfrak{h}$ isomorphic to $\mathfrak{sl}(2,\mathbb{R})$, i.e. generated by an sl_2-triple $\{X_0, Y_0, Z_0\}$, such that $[X_0, Y_0] = 2Y_0$,$[X_0, Z_0] = -2Z_0$, and $[Y_0, Z_0] = X_0$.

Let $H \subset G$ be the group associated to $\mathfrak{h}$. From the lemma above and the fact that G has finite center, H is a finite covering of $PSL(2,\mathbb{R})$. Let us call Σ the H-orbit of p. Since Y_0 vanishes at p, but not X_0 (by Lemma 3.7), Σ is a lightlike surface, homothetic up to finite cover to $(\mathbb{R}^2 \setminus \{0\}, d\theta^2)$. Also, $Y_0 \in \mathfrak{g}_p$ implies that H acts non-properly on Σ. The group $\exp(\mathbb{R}X_0)$ is a Cartan subgroup of H, and by Lemma 2.6, $\exp(\mathbb{R}X_0)$ also acts non-properly. Thus, we can find (q_k) a sequence of Σ converging to $p_0 \in \Sigma$, and a sequence of return times (t_k), such that $h_k.q_k$ converges in Σ, where $h_k = \exp(t_k X_0)$. Because in any finite-dimensional representation of $\mathfrak{sl}(2,\mathbb{R})$, any $\mathbb{R}$-split element is mapped on some $\mathbb{R}$-split element, the Cartan subalgebra $\mathbb{R}X_0$ is contained in a Cartan subalgebra of the ambient algebra $\mathfrak{g}$, say $\mathfrak{a}_0$. Now, we apply the first part of Proposition 3.2 to X_0 and $\mathfrak{a}_0$, and deduce that $W^s_{X_0}$ is isotropic at p_0 (where $W^s_{X_0}$ is defined relatively to $\mathfrak{a}_0$). In particular, Y_0 is isotropic at p_0, and Fact 2.3 then ensures that X_0 is also isotropic at p_0. □

4. Proof of Theorem 1.3

4.1. **Reduction lemma.** The following fact reduces the proof of Theorem 1.3 to the case of non-proper *transitive* actions of semi-simple groups.

Lemma 4.1. (Reduction to the transitive case) *Let G be a semi-simple Lie group with no factor locally isomorphic to $SL(2,\mathbb{R})$, acting faithfully non-properly isometrically on a lightlike manifold. Then there exists a G-orbit which is lightlike and the G-action on it is non-proper, i.e. has non-precompact stabilizers. In fact, the stabilizer subalgebras contain nilpotent elements.*

More precisely, any $p \in M$, for which there exists X such that W^s_X is isotropic at p, has a lightlike orbit on which the action is non-proper.

Proof. We have already seen at the end of the proof of Proposition 3.1, that W^s_X has dimension > 1. If it is isotropic at p, then it contains elements $Y \in W^s_X \cap \mathfrak{g}_p$ vanishing at p. But Y is a nilpotent element in $\mathfrak{g}$, and in particular $Ad(\exp(tY))$ is non-compact, proving that the stabilizer of p is non-compact. Therefore the action of G on the G-orbit O of p is non-proper.

Let us show that O is lightlike. Lemma 3.5 shows that O can not be reduced to p. Also, O can not be 1-dimensional. Indeed, a factor of G should then act faithfully on O, and we already saw that such a factor would be locally isomorphic to $SL(2, \mathbb{R})$.

Suppose now by contradiction that O is Riemannian. Then any vector which is isotropic at p must vanish there, in particular $W^s_X \subset \mathfrak{g}_p$.

Consider $Y \in W^s_X$, and its infinitesimal action on the tangent space of the orbit at p. This action is just $ad(Y) : \mathfrak{g}/\mathfrak{g}_p \longrightarrow \mathfrak{g}/\mathfrak{g}_p$. If O is supposed to be Riemannian, it is at the same time skew symmetric and nilpotent, hence trivial (on $\mathfrak{g}/\mathfrak{g}_p$), which means: $ad(Y)\,(\mathfrak{g}) \subset \mathfrak{g}_p$, for all $Y \in W^s_X$. Now, let us pick $Y_0 \in W^s_X$, and use the Jacobson-Morozov theorem to get an $\mathfrak{sl}(2, \mathbb{R})$-triple (Z_0, X_0, Y_0). Then $ad(Y_0)(Z_0) = X_0$, so that $X_0 \in \mathfrak{g}_p$. But we saw that X_0 is in a Cartan subalgebra of $\mathfrak{g}$, yielding a contradiction with Lemma 3.7.

□

4.2. **Proof in the simple case.** We now give the proof of Theorem 1.3, assuming that the group G is simple with finite center, and the action is transitive and non-proper. The general case of semi-simple groups will be handled in the next section. The proof will be achieved in several steps.

Step 1: There exist $p \in M$ and X in some Cartan subalgebra $\mathfrak{a}$, such that $W^s_X \subset \mathfrak{g}_p$.

Proof. Proposition 3.2 says that for some $p \in M$, both X and W^s_X are isotropic at p. For any Y in W^s_X, the Lie algebra generated by X and Y is isomorphic to the Lie algebra $\mathfrak{aff}(\mathbb{R})$ and acts on the null leaf $\mathcal{N}_p$. Up to isomorphism, there are exactly two actions of $aff(\mathbb{R})$ on a connected 1-manifold:

(1) The usual affine action of $aff(\mathbb{R})$ on the line. Here, a conjugate of X vanishes somewhere.
(2) The non-faithful action, for which Y acts trivially.

We can not be in the first case without contradicting Lemma 3.7, so that only possibility (2) occurs, and thus $W^s_X \subset \mathfrak{g}_p$. □

Step 2: The $\mathbb{R}$-rank of $\mathfrak{g}$ equals 1.

Proof. Suppose the $\mathbb{R}$-rank of $\mathfrak{g} > 1$. Let α be a root such that $\alpha(X) > 0$ and β an adjacent root in the Dynkin diagram, according to the choice of a basis Φ of positive simple roots where, $\gamma \in \Phi \Longrightarrow \gamma(X) \geq 0$. (See [14]).

By definition, $\alpha + \beta$ is also a root and $(\alpha + \beta)(X) > 0$, that is, $\mathfrak{g}_{-\alpha}$ and $\mathfrak{g}_{-(\alpha+\beta)}$ are different and contained in W^s_X.

Let T_α and $T_{\alpha+\beta}$ be the vectors of $\mathfrak{a}$ dual to α and $\alpha + \beta$ respectively. They are linearly independent.

Moreover, $T_\alpha \in [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \subset ad_{\mathfrak{g}}(W^s_X)$, and similarly for $T_{\alpha+\beta}$, By the first step and Lemma 2.4, T_α and $T_{\alpha+\beta}$ are isotropic at p. Hence, there is a non-trivial linear combination of them which vanishes at p. This contradicts Proposition 3.7 claiming that $\mathfrak{a} \cap \mathfrak{g}_p = 0$. Therefore, $\mathfrak{g}$ has rank 1. □

Remark 4.2. *It is exactly here that we need G to be simple!*

Step 3: The Lie algebra $\mathfrak{g}$ is isomorphic to $\mathfrak{o}(1, n)$

Proof. Suppose that $\mathfrak{g}$ is not isomorphic to $\mathfrak{o}(1, n)$, then we have two roots $\alpha, 2\alpha$, such that $\alpha(X) > 0$,

Claim 4.3. The bracket $[\mathfrak{g}_{+2\alpha}, \mathfrak{g}_{-\alpha}] \neq 0$.

Let us continue the proof assuming the claim. Consider a non-zero $Y \in [\mathfrak{g}_{+2\alpha}, \mathfrak{g}_{-\alpha}] \subset \mathfrak{g}_{+\alpha}$. By Lemma 2.4, Y is isotropic at p. Let Θ be the Cartan involution, then $\Theta Y \in W_X^s$, and hence belongs to $\mathfrak{g}_p$, by Step 1. Lemma 2.4 then implies that $[Y, \Theta Y] \in \mathfrak{g}_p$, in particular $\mathfrak{a} \cap \mathfrak{g}_p \neq 0$, which contradicts Lemma 3.7. □

Proof of the claim. First, the rank 1 simple Lie groups of non-compact type are known to be the isometry groups of symmetric spaces of negative curvature. They are the real, complex and quaternionic hyperbolic spaces, together with the hyperbolic Cayley plane. A direct computation can be performed and yields the claim. Let us give another synthetic proof. By contradiction, the sum $\mathfrak{l} = \mathfrak{g}_0 + \mathfrak{g}_{-\alpha} + \mathfrak{g}_{-2\alpha} + \mathfrak{g}_{+2\alpha}$ would be a subalgebra of $\mathfrak{g}$. For the sake of simplicity, let us work with groups instead of algebras. Let L be the group associated to our last subalgebra$\mathfrak{l}$. Clearly, L is not compact. The point is that there is a dichotomy for non-compact connected isometry subgroups of negatively curved symmetric spaces. If they have a non-trivial solvable radical, then they fix a point at infinity, and thus are contained in a parabolic group and in particular have a compact simple (Levi)-part (see [11]). If not the group is semi-simple. It is clear that our L contains a non-compact semi-simple, and therefore by the dichotomy, it is semi-simple. But, once semi-simple, L will have a "symmetric " root decomposition, i.e. the negative of a root is a root, too. Thus, there must exist a non-trivial root space corresponding to α, which contradicts the true definition of $\mathfrak{l}$. □

Step 4: The full isotropic subalgebra is $\mathfrak{s}_p = \mathfrak{a} \oplus \mathfrak{m} \oplus \mathfrak{g}_{-\alpha}$

Proof. Recall that $\mathfrak{m}$ is the Lie algebra of the centralizer of $\mathfrak{a}$ in the maximal compact K. Since $\mathfrak{m} \subset [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$, Lemma 2.4 implies that it is isotropic at p.

On the other hand, if $Y \in \mathfrak{g}_+\alpha$ is isotropic at p, Lemma 2.4 implies that the semi-simple element $[Y, \Theta Y] \in [\mathfrak{g}_{-\alpha}, \mathfrak{g}_\alpha] \subset [\mathfrak{s}_p, W_p^s \cap \mathfrak{g}_p]$ is in the the stabilizer subalgebra of p, which contradicts Lemma 3.7. Therefore, the isotropic subalgebra is exactly $\mathfrak{s}_p = \mathfrak{a} \oplus \mathfrak{m} \oplus \mathfrak{g}_\alpha$. □

Step 5: The full stabilizer subalgebra is $\mathfrak{g}_p = \mathfrak{m} \oplus \mathfrak{g}_\alpha$.

Proof. Let $Z \in \mathfrak{m}$, then it is isotropic at p. Suppose by contradiction that $Z \notin \mathfrak{g}_p$. Then, there exists an element $Z + \lambda X \in \mathfrak{g}_p$, $\lambda \in \mathbb{R}^*$. We let it act on the normal space of the null leaf.

The action of X on $\mathfrak{g}/\mathfrak{s}_p$ is identified to its action on $\mathfrak{g}_{+\alpha}$, by the previous step. In particular, the X-action has non-zero real eigenvalues.

The action of $\mathfrak{m}$ (and in particular Z) on $\mathfrak{g}/\mathfrak{s}_p$ has purely imaginary eigenvalues, since $\mathfrak{m}$ is contained the Lie algebra of a maximal compact group.

On one hand, since X and Z commute (by definition of $\mathfrak{m}$), the action of $Z + \lambda X$ on $\mathfrak{g}/\mathfrak{s}_p$ must have eigenvalues with non-trivial real part.

On the other hand, $Z+\lambda X \in \mathfrak{g}_p$ and acts as a skew symmetric endomorphism on $\mathfrak{g}/\mathfrak{s}_p$, and thus has only purely imaginary eigenvalues: contradiction. This shows that $\mathfrak{m} \subset \mathfrak{g}_p$, but since $\mathfrak{a} \cap \mathfrak{g}_p = 0$, and $\mathfrak{g}_p \subset \mathfrak{s}_p$, which was calculated in the previous space, we get the equality $\mathfrak{g}_p = \mathfrak{m} \oplus \mathfrak{g}_-\alpha$. □

End: Since $\mathfrak{g}$ is isomorphic to $\mathfrak{o}(1,n)$, and the Lie algebra of the stabilizer $\mathfrak{g}_p$ is isomorphic to the Lie algebra of the group of Euclidian motions Euc_n, we conclude that M is a covering of the Lightcone in Minkowski space, which completes the proof of Theorem 1.3 when G is simple.

4.3. **End of the proof.** Thanks to Lemma 4.1, the complete proof of Theoremsimple.transitive reduces to the study of non-proper transitive actions of semi-simple groups with no factor locally isomorphic to $SL(2,\mathbb{R})$. The work made above will be useful thanks to the following reduction lemma:

Lemma 4.4. (Reduction to the simple case) *Let X be in a Cartan subalgebra of $\mathfrak{g}$, such that W^s_X is isotropic at p. Consider the decomposition of $\mathfrak{g}$ in simple factors. Let $\mathfrak{h}$ be such a simple factor, and $H \subset G$ the corresponding group. Suppose X has a non-trivial projection on $\mathfrak{h}$. Then the H-orbit is non-proper and lightlike.*

Proof. Write $\mathfrak{g} = \mathfrak{h}_1 \oplus ... \oplus \mathfrak{h}_s$, where the $\mathfrak{h}_i$'s are the simple factors of $\mathfrak{g}$, and call X_i the projection of X on $\mathfrak{h}_i$. If $W^s_{X_i,\mathfrak{h}_i}$ denotes the stable space of X_i relatively to $\mathfrak{h}_i$, it is straightforward to check that $X^s_X = W^s_{X_1,\mathfrak{h}_1} \oplus ... \oplus W^s_{X_s,\mathfrak{h}_s}$. In particular, if W^s_X is isotropic at p and $\mathfrak{h}$ is a simple factor on which X has a non-trivial projection X', then $W^s_{X',\mathfrak{h}}$ is non-trivial and isotropic at p. We infer from Corollary 4.1 that the H-orbit of p is lightlike and the action of H on it is non-proper. □

By this lemma, there is a simple factor H of G having a lightlike non-proper orbit $H.p$. It follows from the previous section that H is locally isomorphic to $O(1,n)$, $n \geq 3$, and $H.p$ is homothetic to Co^n. There is a semi-simple group H' such that G is a finite quotient of $H \times H'$. This product still acts locally faithfully on M, so that we will assume $G = H \times H'$ in the following. Consider $O = G.p$ the G-orbit containing this $H.p$. The remaining part of Theorem 1.3 is the geometric description of O: it is a direct metric product $Hp \times H'p$ (up to a finite cover). This is the containt of the following lemma, which will be also useful when dealing with groups having factors locally isomorphic to $SL(2,\mathbb{R})$.

Proposition 4.5. *Let G be a semi-simple Lie group acting locally faithfully transitively and non properly on a lightlike manifold (M,h). We assume that $G = H \times H'$, where H is isomorphic to $O(1,d)$, $d \geq 2$, and H' is semi-simple. We assume that for some $p \in M$, the orbit $H.p$ is homothetic to Co^d (resp. Co^2 or Co^1) if $d \geq 3$ (resp. $d = 2$). Then M is a metric product $M = Co^m \times N$, where N is an H'-homogeneous Riemannian manifold.*

Proof. M is naturally foliated by lightcones $\mathcal{H}_x = H.x$. This foliation is G-invariant: if $g \in G$, then, $g\mathcal{H}_x = gH.x = Hg.x = \mathcal{H}_{gx}$, since H is normal in G.

1) We first prove that $H'.p$ is Riemannian. If it is not the case, it contains the null leaf $\mathcal{N}_p$. If $p' \neq p$ is a point of $\mathcal{N}_p$, then there exists $h' \in H'$ such that $h'.p = p'$. Also, h' maps the null line of $H.p$ passing through p onto the null line of $H.p'$ passing through p', which means that h' preserves $\mathcal{N}_p$ and acts non-trivially on it. But since p' is a point of $\mathcal{N}_p \subset H.p$, h' maps $H.p$ on itself. Thus, h' is an isometry of the cone $H.p$, commuting with the action of H.

Lemma 4.6. *Let h' be an isometry of the cone Co^d, $d \geq 2$ (resp. of Co^1), commuting with the action of $O(1,d)$ (resp. $O(1,2)$). Then h' is the identity map of Co^d (resp. Co^1).*

Proof. We begin with the case of Co^1. An isometry of Co^1 is just a diffeomorphism of $\mathbf{S}^1$. If such a diffeomorphism commutes with the projective action of $O(1,2)$ on $\mathbf{S}^1$, it must fix, in particular, all the fixed points of parabolic elements in $O(1,2)$. But the set of these fixed points is precisely $\mathbf{S}^1$, so that we are done.

Now, in higher dimension, we saw that writing Co^d as $\mathbb{R} \times \mathbf{S}^{d-1}$ with the metric $0 \oplus g_{\mathbf{S}^{d-1}}$, the isometry h' is of the form: $(t,x) \mapsto (t - \mu(x), \phi(x))$. Here ϕ is a conformal transformation of $\mathbf{S}^{d-1}$ satisfying $\phi^* g_{\mathbf{S}^{d-1}} = e^{2\mu} g_{\mathbf{S}^{d-1}}$. Now, if h' commutes with the action of $O(1,d)$, it must leave invariant any line of fixed points of parabolic elements in $O(1,d)$. This yields $\phi(x) = x$, and finally $\mu(x) = 0$.

□

The lemma implies that h' acts identically on $H.p$, a contradiction with the fact that it acts non-trivially on $\mathcal{N}_p$.

2) Let S be the isotropy group of p in H. Since H and H' commute, S acts trivially on $H'.p$: $s(h'.p) = h's.p = h'.p$. In particular, S will act trivially on $T_p(H'.p) \cap T_p(H.p)$. But $T_p(\mathcal{N}_p)$ is the only subspace of $T_p(H_p)$ on which the action is trivial, and since we have already seen that $T_p(H'.p)$ must be transverse to $T_p(\mathcal{N}_p)$, we get $T_p(H'.p) \cap T_p(H.p) = \{0\}$. Now, there is a S-invariant splitting $T_pM = T_p(H'.p) \oplus T_p(\mathcal{N}_p) \oplus E$, where E is a Riemannian subspace of $T_p(H.p)$, on which S acts irreducibly by the

standard action of $O(n-1)$ on $\mathbb{R}^{n-1}$. Let us call F the orthogonal of $T_p(H.p)$ in T_pM. This space is transverse to E, so that F is the graph of a linear map $A : T_p(H'.p) \oplus T_p(\mathcal{N}_p) \to E$. This map A intertwines the trivial action of S on $T_p(H'.p) \oplus T_p(\mathcal{N}_p)$ with the irreducible one on E, so that $A = 0$, and $F = T_p(H'.p) \oplus T_p(\mathcal{N}_p)$. As a consequence, the sum $T_p(H'.p) \oplus T_p(H.p)$ is orthogonal for the metric h_p, and by homogeneity of M, this remains true at every point of M. □

4.4. **Proof of corollary 1.4.** Here, we assume that G is semi-simple, non-compact, with finite center. The group G acts transitively and non-properly on a lightlike manifold (M, h). Looking at a finite cover of G if necessary, we assume that $G = H_1 \times ... \times H_s$, where each H_i is a simple group with finite center. For $p \in M$, and every $i = 1, ..., s$, we call G_p^i the projection of the isotropy group G_p on H_i, and H_p^i the intersection $G_p \cap H_i$. Each H_p^i is a normal subgroup of G_p. Since G_p is non-compact, some G_p^i has non-compact closure; for example $i = 1$. Performing a Cartan decomposition of a sequence (g_k) in G_p tending to infinity, and using Proposition 3.2, we get X_1 in a Cartan subalgebra of $\mathfrak{h}_1$, and some $p' \in M$ such that W_{X_1} is isotropic at p'. If H_1 is not locally isomorphic to $SL(2, \mathbb{R})$, we get that W_{X_1} has dimension > 1, and thus $H_1.p'$ is lightlike and carries a non-proper action of H_1. By the previous study, H_1 is isomorphic to $O(1, n)$, $n \geq 3$, and $H_1.p'$ is homothetic to Co^n. We can then apply Proposition 4.5 to conclude.

We are left with the case where H_1 is a finite cover of $PSL(2, \mathbb{R})$, and G_p^1 does not have compact closure. We claim that the orbit $H_1.p$ can not have dimension 3. Indeed, let $< >_p$ the pullback of h_p in the Lie algebra $\mathfrak{h}_1$. Let $g \in G_p$, and g_j the projection of g on H_j. Since D_pg leaves h_p invariant, and the H_j's commute, we get that $< >_p$ is $Ad(g_1)$-invariant. But $< >_p$ is either Riemannian, or lightlike. In both cases, we saw (see the proof of Proposition 2.1) that the subgroup $S \subset H_1$ such that $Ad(S)$ preserves $< >_p$ is compact, contradicting the fact that G_p^1 does not have compact closure.

Now, if $H_1.p$ is of dimension 2 and Riemannian, H_p^1 is a maximal compact subgroup $K \subset H_1$. Now, since H_p^1 is normal in G_p, we get that K is normal in G_p^1, what yields $G_p^1 = K$, and a new contradiction.

We conclude that $H_1.p$ is either 1-dimensional and lightlike, or 2-dimensional and lightlike. It follows from Proposition 2.1 that $H_1.p$ is homothetic to a cone Co^1 or Co^2. We then get the conclusion thanks to Proposition 4.5. □

5. Proof of theorem 1.5

Here, we assume that G is simple with finite center, and acts locally faithfully by isometries of a compact lightlike manifold (M, h).

We first assume, by contradiction, that G is not locally isomorphic to $PSL(2, \mathbb{R})$. By compactness , every sequence of M is non-escaping. It follows from Proposition 3.2 that for every X in a Cartan subalgebra of $\mathfrak{g}$, W_X^s is isotropic at every $p \in M$. Thus, using the last point of Lemma

4.1, and the conclusions of Corollary 1.4, we get that G is locally isomorphic to $O(1,n)$, and any G-orbit is homothetic to Co^n, $n \geq 3$. Let us call K a maximal compact subgroup of G, and let K_0 the stabilizer in K of a given point $p_0 \in M$. As we saw it in the proof of Lemma 3.5, the compact group K_0 preserves a Riemannian metric on M. Since any Riemannian isometry can be linearized around any fixed point thanks to the exponential map, it is not difficult to prove that the set of fixed points of K_0 is a closed submanifold of M, that we call M_0. We know explicitely the action of K on Co^n, and observe that every orbit of K is of Riemannian type. Let $S(\mathfrak{k}/\mathfrak{k}_0)$ denote the set of euclidean scalar products on $\mathfrak{k}/\mathfrak{k}_0$. There is a continuous map $\mu : M_0 \to S(\mathfrak{k}/\mathfrak{k}_0)$ defined in the following way: if X and Y are two vectors of $\mathfrak{k}$, and $\overline{X}$ and $\overline{Y}$ are their projections on $\mathfrak{k}/\mathfrak{k}_0$, then $\mu(p)(\overline{X},\overline{Y}) = h_p(X(p), Y(p))$. Now, on $G.p_0$, there is a 1-parameter flow of homotheties h^t, which transforms $h_{|G.p_0}$ into $e^{2t}h_{|G.p_0}$, and commutes with the action of K (in particular, it leaves $M_0 \cap G.p_0$ invariant). From this, it follows that $\mu(h^t.p_0) = e^{2t}\mu(p_0)$. Now, by compactness of M_0, there is a sequence (t_k) tending to $+\infty$ such that $h^{t_k}.p_0$ tends to $p_\infty \in M_0$. We should get by continuity of μ: $\lim_{k\to+\infty} e^{2t_k}\mu(p_0) = \mu(p_\infty)$, which yields the desired contradiction.

It remains to understand what happens if G has finite center, and is locally isomorphic to $PSL(2,\mathbb{R})$. Let us fix X, Y, Z a standard basis of $\mathfrak{g}$: $[Y,Z] = X$,$[X,Y] = 2Y$, and $[X,Z] = -2Z$. It follows from Proposition 3.2 that Y and Z are isotropic at every $p \in M$. As a consequence, at any $p \in M$, a nontrivial linear combination of Y and Z has to vanish, so that all the orbits of G are lightlike and have dimension at most 2. If there is a 2-dimensional orbit $G.p_0$, Proposition 2.1 ensures that it is homothetic to $\mathbb{R}^2 \setminus \{0\}$ endowed with the metric $d\theta^2$ (namely Co^2). We get a contradiction exactly as above, using the action of a maximal compact group and the homothetic flow on Co^2 (here $\mathfrak{k}_0 = 0$).

We conclude that every G orbit is 1-dimensional and lightlike. Since G has finite center, these orbits are finite coverings of the circle, hence closed. □

6. Proof of Theorem 1.6

Let us first summarize results on Lorentz dynamics in the following statement, fully proved in [9], but early partially proved for instance in [1, 2, 4, 16].

Theorem 6.1. *Let G be a semi-simple group with finite center, no compact factor and no local factor isomorphic to $SL(2,\mathbb{R})$, acting isometrically non-properly on a Lorentz manifold M. Then, up to a finite cover, G has a factor G' isomorphic to $O(1,n)$ or $O(2,n)$ and having some orbit homothetic to dS_n or AdS_n.*

Most developments along the article, in particular Proposition 3.2, do not explicitly involve the lightlike nature of the ambient metric, and apply equally to the Lorentz case, and by the way to the general sub-Lorentz case. This allows one to find a non-proper G-orbit O, i.e. with a stabilizer algebra containing nilpotent elements (see the end of proof of Proposition 3.1). One checks easily that O can not be Riemannian. If O is Lorentz, then, apply Theorem 6.1 (in the homogeneous case), and if it is lightlike, then apply Theorem 1.3. □

6.0.1. *Some remaining questions.* The results of [9] are stronger than the statement of Theorem 6.1, since they contain a detailed geometric description of the Lorentz manifold M (a warped product structure...). This is the missing part of Theorem 1.3 in the lightlike non-homogeneous case and Theorem 1.6 in the sub-Lorentz case. In particular, in this last sub-Lorentz situation, it remains to see whether the manifold is or not pure, i.e. everywhere lightlike, or everywhere Lorentz?

References

[1] S. Adams: Orbit nonproper actions on Lorentz manifolds, Geom. Funct. Anal. **11** (2001), no. 2, 201-243.

[2] S. Adams: Dynamics of semisimple Lie groups on Lorentz manifolds, Geom. Ded. **105** (2004), 1-12.

[3] M. Akivis, V. Goldberg, On some methods of construction of invariant normalizations of lightlike hypersurfaces. Differential Geom. Appl. 12 (2000), no. 2, 121-143.

[4] A. Arouche, M. Deffaf, A. Zeghib , On Lorentz dynamics: From group actions to warped products via homogeneous espaces. Trans. Amer. Math. Soc. 359 (2007) 1253–1263

[5] E. Bekkara, C. Frances, A. Zeghib, On lightlike geometry: isometric actions, and rigidity aspects. C. R. Math. Acad. Sci. Paris 343 (2006), no. 5, 317–321.

[6] F. Bonsante, Flat spacetimes with compact hyperbolic Cauchy surfaces. J. Differential Geom. 69 (2005) 441–521.

[7] Y. Carrière, Flots riemanniens. Astérisque No. 116 (1984), 31–52.

[8] G. D'Ambra, M. Gromov, Lectures on transformation groups: geometry and dynamics. Surveys in differential geometry (Cambridge, MA, 1990), 19–111, Lehigh Univ., Bethlehem, PA, 199.

[9] M. Deffaf, K. Melnick, A. Zeghib, Actions of noncompact semi-simple groups on Lorentz manifolds, to appear in GAFA. ArXiv math.DS/0601165.

[10] K. Duggal, A. Bejancu, Lightlike submanifolds of semi-Riemannian manifolds and applications. Mathematics and its Applications, 364. Kluwer Academic Publishers Group, Dordrecht, 1996.

[11] P Eberlein: Geometry of Nonpositively Curved Manifolds, University of Chicago Press, Chicago, 1996.

[12] F. Friedlander, The wave equation on a curved space-time. Cambridge Monographs on Mathematical Physics, No. 2. Cambridge University Press, Cambridge-New York-Melbourne, 1975.

[13] S. Hawking, G. Ellis, The large scale structure of space-time. Cambridge Monographs on Mathematical Physics, No. 1. Cambridge University Press, London-New York, 1973.

[14] A. Knapp, Lie groups beyond an introduction. Second edition. Progress in Mathematics, 140. Birkhuser Boston, Inc., Boston, MA, 2002

[15] T. Kobayashi, T. Yoshino, Compact Clifford-Klein form of symmetric spaces - revisited, Pure Appl. Math. Q., **1(3)**, (2005) 591-663.

[16] N. Kowalsky, Noncompact simple automorphism groups of Lorentz manifolds. Ann. Math. 144 (1997), 611-640.

[17] D. Kupeli, Singular semi-Riemannian geometry. With the collaboration of Eduardo Garca-Ro on Part III. Mathematics and its Applications, 366. Kluwer Academic Publishers Group, Dordrecht, 1996.

[18] T. Miernowski, Thesis, école normale supérieure de Lyon, 2005.

[19] P. Molino, Riemannian foliations. Progress in Mathematics, 73. Birkhuser Boston, Inc., Boston, MA, 1988
[20] P. Chruściel, On rigidity of analytic black holes. Comm. Math. Phys. 189 (1997) 1-7.
[21] I. Robinson, A. Trautman, Integrable optical geometry, *Lett. Math. Phys.* **10**, 179–182 (1985).
[22] A. Zeghib, Geodesic foliations in Lorentz 3-manifolds. Comment. Math. Helv. 74 (1999) 1–21.
[23] A. Zeghib, Isometry groups and geodesic foliations of Lorentz manifolds. II. Geometry of analytic Lorentz manifolds with large isometry groups. Geom. Funct. Anal. 9 (1999), 823–854.

⋆ ENSET-Oran, Algeria

E-mail address: esmaa.bekkara@gmail.com

† Laboratoire de Mathématiques, Univ. Paris Sud.

E-mail address: charles.frances@math.u-psud.fr

‡ CNRS, UMPA, École Normale Supérieure de Lyon.

E-mail address: zeghib@umpa.ens-lyon.fr